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NILPOTENT-BY-FINITE GROUPS WITH ISOMORPHIC FINITE QUOTIENTS

BY

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ABSTRACT. Let $\mathfrak{F}(G)$ denote the set of isomorphism classes of finite homomorphic images of a group G. We say that groups G and H have isomorphic finite quotients if $\mathfrak{F}(G) = \mathfrak{F}(H)$. Let $\mathfrak{N}$ denote the class of finite extensions of finitely generated nilpotent groups. In this paper we show that if G is in $\mathfrak{N}$, then the groups H in $\mathfrak{N}$ for which $\mathfrak{F}(G) = \mathfrak{F}(H)$ lie in only finitely many isomorphism classes.

Let $\mathfrak{F}(G)$ denote the set of isomorphism classes of finite homomorphic images of the group G. We say that groups G and G_1 have isomorphic finite quotients if $\mathfrak{F}(G) = \mathfrak{F}(G_1)$. In [14] it was shown that if G is a finitely generated nilpotent group, then the finitely generated nilpotent groups G_1, for which $\mathfrak{F}(G) = \mathfrak{F}(G_1)$, lie in only finitely many isomorphism classes. In this paper we prove the following extension of that result:

Main Theorem. *Let $\mathfrak{N}$ denote the class of all finite extensions of finitely generated nilpotent groups. If G is in $\mathfrak{N}$, then the groups G_1 in $\mathfrak{N}$, for which $\mathfrak{F}(G) = \mathfrak{F}(G_1)$, lie in only finitely many isomorphism classes.*

It should perhaps be noted that the class $\mathfrak{N}$ contains all known examples of nonisomorphic polycyclic groups with isomorphic finite quotients (Remeslemnikov [15], Dyer [8], Higman (unpublished) and Brigham [7]).(1)

The proof follows the general plan of the previous paper, using many of the constructions and results of that paper, as well as new techniques motivated in part by the theorem of Auslander and Baumslag [2] and its proof [3]. The notations in [14] will be preserved except that torsion-free finitely generated nilpotent groups will be denoted $\mathfrak{T}$-groups after P. Hall [10].

If G is a group we may form the profinite topology on G by taking as neighborhood basis of the identity the subgroups of finite index in G. We will denote by $\hat{G}$ the completion of G in this topology. If G is an M-group (polycyclic-by-finite, see [16]), G is residually finite so the topology is Hausdorff and G is canonically included in $\hat{G}$. Also the topology coincides with the topology for which the sub-

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(1) Added in proof. Remeslennikov (Siberian Math. J. 12 (1971), 791–792) has recently given examples which are not in $\mathfrak{N}$.

groups $G^m = \mathrm{gp}\{g^m \mid g \in G\}$, for all positive integers m, form a neighborhood basis for the identity. Since G/G^m is the largest quotient of G of exponent m, $\mathfrak{F}(G) = \mathfrak{F}(G_1)$ implies that G/G^m must be isomorphic to G_1/G_1^m for all positive integers m. Conversely, every finite quotient of G is a quotient of G/G^m for some m. Thus if G/G^m is isomorphic to G_1/G_1^m for each m, we must have $\mathfrak{F}(G) = \mathfrak{F}(G_1)$.

If m divides n we have a natural homomorphism ϕ_{nm}: $G/G^n \rightarrow G/G^m$. The groups G/G^m together with the homomorphisms ϕ_{nm} form an inverse system whose inverse limit is $\hat{G}$. $\hat{G}/\hat{G}^m$ is isomorphic to G/G^m for each m, so that if $\hat{G}$ is isomorphic to $\hat{G}_1$, we must have G/G^m isomorphic to G_1/G_1^m for all positive integers m. If G/G^m is isomorphic to G_1/G_1^m for all m, we may show, as in the proof of Lemma 1.2 of [14], that we may choose isomorphisms θ_m: $G/G^m \rightarrow G_1/G_1^m$ which are compatible with the maps ϕ_{nm} and ϕ^1_{nm}. The isomorphisms θ_m form an isomorphism of the inverse systems and thus induce an isomorphism of the inverse limits. That is, we must have $\hat{G}$ isomorphic to $\hat{G}_1$. The preceding paragraphs prove the following analogue of Lemma 1.2 of [14]:

Proposition 1. *Let G and G_1 be M-groups. Then $\mathfrak{F}(G) = \mathfrak{F}(G_1)$ if and only if $\hat{G}$ is isomorphic to $\hat{G}_1$.*

If H is a subgroup of the M-group G, then the profinite topology on H is the same as the topology induced by the profinite topology of G (Theorem 20B of [9]). A simple calculation with Cauchy sequences shows that the completion $\hat{H}$ of H can be considered as a subgroup of $\hat{G}$. If, in addition, H is normal in G then $\hat{H}$ is normal in $\hat{G}$ and $(G/H)^\wedge$ is isomorphic to $\hat{G}/\hat{H}$.

Suppose now that G is in $\mathfrak{N}$ and let H be the maximal normal nilpotent subgroup of G and let π: $G \rightarrow G/H = F$ be the canonical projection. Then $\hat{H}$ is a normal nilpotent subgroup of $\hat{G}$ and π extends to $\hat{\pi}$: $\hat{G} \rightarrow F$ with kernel $\hat{H}$ (since $F = \hat{F}$).

Lemma 2. *With G and H as above, the maximal normal nilpotent subgroup of $\hat{G}$ is $\hat{H}$.*

Proof. Let M^* be the maximal normal nilpotent subgroup of $\hat{G}$; then $M^* \geq \hat{H}$ and $\hat{\pi}(M^*)$ is nilpotent. Let $M = \pi^{-1}(\hat{\pi}(M^*)) \leq G$. Then M is a normal subgroup of G containing H such that $\hat{M} = M^*$. M is polycyclic, since it is the extension of H by the nilpotent group $\hat{\pi}(M^*)$. Since $\hat{M} = M^*$ is nilpotent, every finite quotient of M is nilpotent. By Theorem 3.24 of [11], M must be nilpotent so we must have $M \leq H$. Thus $M = H$ and $\hat{H} = \hat{M} = M^*$.

Suppose now that G_1 and G_2 are in $\mathfrak{N}$ and that their respective maximal normal nilpotent subgroups are H_1 and H_2, and suppose that G_1 and G_2 have isomorphic finite quotients. Any isomorphism ϕ: $\hat{G}_1 \rightarrow \hat{G}_2$ must by Lemma 2 send $\hat{H}_1$ onto $\hat{H}_2$. It follows that the respective factor groups G_1/H_1 and G_2/H_2 must be isomorphic and by Proposition 3.5 of [14], ϕ must take the torsion subgroup τH_1 of H_1 onto the torsion subgroup τH_2 of H_2. (Recall that $(\tau H)\hat{} = \tau\hat{H}$.) Since the torsion subgroups are fully invariant in their respective H_i, they are normal in G_i. Thus we may form $G_i^* = G_i/\tau H_i$ with the normal subgroup $H_i^* = H_i/\tau H_i$ for $i = 1, 2$. The isomorphism ϕ induces an isomorphism $\phi^*: \hat{G}_1^* \to \hat{G}_2^*$ which takes $\hat{H}_1^*$ onto $\hat{H}_2^*$.

A finite extension G of a $\mathcal{T}$-group H (finitely generated torsion-free nilpotent) will be called a $\mathcal{T}F$-group. A $\mathcal{T}F$-group should be considered as the pair (G, H), i.e. a $\mathcal{T}F$-group G is supposed to have a distinguished normal $\mathcal{T}$-subgroup H. If two $\mathcal{T}F$-groups G_1 and G_2 with respective $\mathcal{T}$-subgroups H_1 and H_2 are isomorphic by an isomorphism ϕ which takes H_1 onto H_2, we will say that G_1 and G_2 are Z-isomorphic. If $\hat{G}_1$ and $\hat{G}_2$ are isomorphic via an isomorphism which takes $\hat{H}_1$ onto $\hat{H}_2$, we will say that G_1 and G_2 are $\hat{}$-isomorphic. In this terminology what we have shown in the previous paragraph is that if G_1 and G_2 are in $\mathfrak{N}$, with maximal normal nilpotent subgroups H_1 and H_2 respectively, and if G_1 and G_2 have isomorphic finite quotients, then (G_1^*, H_1^*) and (G_2^*, H_2^*) are $\hat{}$-isomorphic.

The proof of Theorem 3.6 of [14] shows that there are only finitely many isomorphism classes of extensions of a finite group by a finitely presented group. Applying this to τH and G^*, we find that there are only finitely many isomorphism classes of extensions of τH by G^*. If G_1 and G_2 have isomorphic finite quotients, τH_1 and τH_2 are isomorphic and G_1^* and G_2^* are $\hat{}$-isomorphic. The preceding statements thus allow us to make the following:

First reduction. It is sufficient to show that the number of Z-isomorphism classes of $\mathcal{T}F$-groups which are $\hat{}$-isomorphic to a given $\mathcal{T}F$-group G is finite.

We now describe the construction of various groups which will be used in the proof. Suppose G is an extension of the $\mathcal{T}$-group H by the finite group F. Let $f_1, \cdots, f_k$ be coset representatives for H in G. Denote by α_i the automorphism of H given by conjugation by f_i and suppose that $f_i f_j = f_{ij} h_{ij}$. That is, the coset representative of $f_i f_j$ is f_{ij}. Then every element of G may be written uniquely in the form $f_i h$ for some i and some h in H. Multiplication is given by

$$(1) \qquad (f_i h)(f_j k) = f_{ij} h_{ij} \alpha_j(h) k.$$

Now let XH denote any of the groups $Z_p H$, $Q_p H$, QH, $\hat{H} = \Pi_{p \in P} Z_p H$, $Q\hat{H}$ (the Malcev completion of $\hat{H}$) or $\Pi H = \Pi_{p \in P} Q_p H$ (see [14] for definitions). Any automorphism of H extends uniquely to an automorphism of XH. Let XG be the group whose elements are (formal) products $f_i h$ with f_i as above and h an element of XH. We define the multiplication by formula (1) where α_j now denotes the extension of the original α_j to XH. Thus we have groups $Z_p G$, $Q_p G$, QG, $\hat{G}$, $Q\hat{G}$ and ΠG respectively. Note that the newly defined $\hat{G}$ is the same as the previously defined profinite completion. Also it should be noted that the group XG depends (in general) on both G and H, a problem which is circumvented by our convention that a $\mathfrak{T}F$-group is a pair (G, H). We shall say that two $\mathfrak{T}F$-groups G_1 and G_2 are X-isomorphic where X is Z, Z_p, Q_p, Q, $\hat{}$, Π or $\hat{Q}$ if XG_1 is isomorphic to XG_2 by an isomorphism which takes XH_1 onto XH_2.

Now suppose G_1 and G_2 are $\mathfrak{T}F$-groups and $\alpha: \hat{G}_1 \to \hat{G}_2$ is an isomorphism which takes $\hat{H}_1$ onto $\hat{H}_2$. $\hat{H}_i = \Pi_{p \in P} Z_p H_i$ and $\alpha(Z_p H_1) = Z_p H_2$. Thus α sends $\Pi_{q \neq p} Z_q H_1$ to $\Pi_{q \neq p} Z_q H_2$ and induces an isomorphism α_p of the quotient $Z_p G_1 = \hat{G}_1/\Pi_{q \neq p} Z_q H_1$ with the quotient $Z_p G_2 = \hat{G}_2/\Pi_{q \neq p} Z_q H_2$. Thus if G_1 and G_2 are $\hat{}$-isomorphic, they must be Z_p-isomorphic for each prime p. Since isomorphisms from $Z_p H_1$ to $Z_p H_2$ extend (uniquely) to isomorphisms from $Q_p H_1$ to $Q_p H_2$, G_1 and G_2 must also be Q_p-isomorphic for each prime p. Since QH is the largest divisible subgroup of QG, if QG_1 and QG_2 are isomorphic, the isomorphism must take QH_1 onto QH_2. Thus if QG_1 and QG_2 are isomorphic, G_1 and G_2 are Q-isomorphic. We now prove the following extension of Theorem 3.1 of [14]:

Theorem 3. *The number of Q-isomorphism classes of $\mathfrak{T}F$-groups which are $\hat{}$-isomorphic to a given group G is finite.*

Proof. We begin with a lemma which will be used several times in the sequel. Since the result is essentially known (see [12, Theorem 6]) and the proof is of a different character from the remainder of the paper, the proof has been relegated to an appendix.

Lemma 4. *If H is a $\mathfrak{T}$-group, then there is a unipotent algebraic matric group $\mathfrak{H}$ and an isomorphism $\phi_Z: H \to \mathfrak{H}_Z$ which extends to isomorphisms $\phi_Q: QH \to \mathfrak{H}_Q$, $\phi_{Z_p}: Z_pH \to \mathfrak{H}_{Z_p}$ and $\phi_{Q_p}: Q_pH \to \mathfrak{H}_{Q_p}$ any two of which agree where they are both defined. If α is an automorphism of H, then α induces a unique algebraic automorphism of $\mathfrak{H}$.*

By Proposition 2.3 of [3], QG and Q_pG are split extensions of QH and Q_pH respectively. Let E be any complement of QH in QG ($E \cap QH = 1$ and $QG = E(QH)$). E is also a complement of Q_pH in Q_pG. The elements of E induce algebraic automorphisms of $\mathfrak{H}$ (of Lemma 4) so we may construct the semidirect product $\mathfrak{G}$ of $\mathfrak{H}$ and E, in such a way that $\mathfrak{G}_Q$ is isomorphic to QG and $\mathfrak{G}_{Q_p}$ is isomorphic to Q_pG for each prime p. Let G_1 be another $\mathfrak{T}F$-group and $\mathfrak{G}^1$ the corresponding algebraic group. If G and G_1 are $\hat{}$-isomorphic, by the discussion prior to the statement of the Theorem and the definition of $\mathfrak{G}$ and $\mathfrak{G}^1$, $\mathfrak{G}_{Q_p}$ is isomorphic to $\mathfrak{G}^1_{Q_p}$ for all primes p. Also G and G_1 are Q-isomorphic if and only if $\mathfrak{G}_Q$ is isomorphic to $\mathfrak{G}^1_Q$. Thus Q-isomorphism classes of $\mathfrak{T}F$-groups which are

$\hat{}$-isomorphic to G can be of no greater number than Q-forms of affine algebraic groups which are Q_p-isomorphic to $\mathfrak{G}$ for all primes p. By Theorem 7.11 of [6], this latter number is finite.

We will say that a $\mathfrak{T}F$-group G_1 is in the genus of G if G and G_1 are both Q-isomorphic and $\hat{}$-isomorphic. Theorem 3 allows us to make the following:

Second reduction. It is sufficient to show that the number of Z-isomorphism classes of $\mathfrak{T}F$-groups in the genus of a given $\mathfrak{T}F$-group G is finite.

Now we let $\mathcal{G}_A$ denote the subgroup of $\mathrm{Aut}(\Pi G)$ consisting of those automorphisms α which stabilize Q_pH for all primes p and which stabilize Z_pH for all but a finite number of primes p. Let $\mathcal{G}_A^\infty$ denote the subgroup of $\mathcal{G}_A$ which stabilizes Z_pH for all primes p. If β is an automorphism of QG, it must stabilize QH. If $f_1, \cdots, f_k$ are coset representatives of H in G, the homomorphism β is determined by $\beta(f_i)$ and the restriction β' of β to QH. β' may be extended uniquely to an automorphism of Q_pH for each prime p. This extension is compatible with the action of the elements f, i.e. $\beta'(h^f) = \beta'(h)^{\beta(f)}$. Thus β' extends to an automorphism of ΠH and we may define an extension of β to ΠG by $\beta(f_i h) = \beta(f_i)\beta'(h)$ for h in ΠH. By Lemma 1.9 of [14], the extension of β to Q_pH stabilizes Z_pH for all but a finite number of primes p. Thus the extension of β is an element of $\mathcal{G}_A$. The subgroup of $\mathcal{G}_A$ consisting of elements obtained in this way from $\mathrm{Aut}(QG)$ will be denoted $\mathcal{G}_Q$. The following proposition allows us to make the next reduction. The proof is the same as the proof of Proposition 3.2 of [14] and will be omitted.

Proposition 5. *The Z-isomorphism classes of $\mathfrak{T}F$-groups in the genus of the $\mathfrak{T}F$-group G are in one-to-one correspondence with a subset of the set of double cosets* $\mathcal{G}_A^\infty \backslash \mathcal{G}_A / \mathcal{G}_Q$.

Third reduction. It is sufficient to show that the number of double cosets $\mathcal{G}_A^\infty \backslash \mathcal{G}_A / \mathcal{G}_Q$ is finite.

From this point the proof in outline is close to the proof of Theorem 4.7 of [3]. We begin with the following:

Fourth reduction. We may assume that G is a split extension of H and that H is lattice nilpotent (see [1] or [13]).

Proof. We replace G by G_1 which is obtained as follows. Let $f_1, \cdots, f_k$ be coset representatives for H in G. Since QG is a split extension of QH, there are elements $a_1, \cdots, a_k$ in QH such that $f_1a_1, \cdots, f_ka_k$ form a group isomorphic with the quotient $F = G/H$. By definition of QH, there is an integer m such that all $a_1^m, \cdots, a_k^m$ are in H. We let $H' = \mathrm{gp}\{x \in QH \mid x^m \in H\}$. Then H' contains H as a subgroup of finite index, and every automorphism of H extends to H' in the obvious manner. Let H_1 be the lattice nilpotent envelope of H' (see [1]). Then

H_1 is a finite extension of H' and every automorphism of H' extends to H_1. Thus we may construct the group G_1 with elements of the form $f_i h$ for some i and some h in H_1. The multiplication is defined by formula (1) where α_j is the extension to H_1 of the automorphism of H given by conjugation with the element f_j. G_1 is a split extension of H_1, since H_1 contains H' and also H_1 is lattice nilpotent.

Denote the various $\mathcal{G}$ corresponding to G_1 by $'\mathcal{G}_A^\infty$, $'\mathcal{G}_A$, $'\mathcal{G}_Q$. We must show that the number of double cosets $\mathcal{G}_A^\infty \backslash \mathcal{G}_A / \mathcal{G}_Q$ is finite if and only if the number of double cosets $'\mathcal{G}_A^\infty \backslash {}'\mathcal{G}_A / {}'\mathcal{G}_Q$ is finite. Since $QH = QH_1$, $QG = QG_1$ and thus $\Pi G = \Pi G_1$. By Lemma 1.8 of [14], $Z_p H = Z_p H_1$ for all primes p except those in a finite set V. Suppose an element α of $\mathrm{Aut}(\Pi G) = \mathrm{Aut}(\Pi G_1)$ is in $\mathcal{G}_A$. Then $\alpha(Z_p H) = Z_p H$ for all primes except those in a finite set W. Thus $\alpha(Z_p H_1) = Z_p H_1$ for all primes except perhaps those in the finite set $V \cup W$, so α is in $'\mathcal{G}_A$. Similarly α in $'\mathcal{G}_A$ implies α in $\mathcal{G}_A$ so $\mathcal{G}_A = {}'\mathcal{G}_A$. Since $QG = QG_1$, $\mathcal{G}_Q = {}'\mathcal{G}_Q$. Using Proposition 1.14 of [14] as in the proof of Theorem 3.3 of [14], we may construct a subgroup K which is of finite index in each of $\mathcal{G}_A^\infty$ and $'\mathcal{G}_A^\infty$. Then $|\mathcal{G}_A^\infty \backslash \mathcal{G}_A / \mathcal{G}_Q|$ is finite if and only if $|K \backslash \mathcal{G}_A / \mathcal{G}_Q|$ is finite if and only if $|{}'\mathcal{G}_A^\infty \backslash {}'\mathcal{G}_A / {}'\mathcal{G}_Q|$ is finite, as required.

Assuming now that G is a split extension, we recall that a subgroup E of G is a complement of H if $H \cap E = 1$ and $HE = G$. If E is a complement of H in G, then E is a complement of $\hat{H}$ in $\hat{G}$, of $Q\hat{H}$ in $Q\hat{G}$, of QH in QG and of ΠH in ΠG. If E_1 is another complement of XH in XG, we will say that E is equivalent to E_1 (in XG) if there is an element h in XH such that $h^{-1}Eh = E_1$.

Lemma 6. *Let G be a split $\mathcal{J}F$-group, the extension of the $\mathcal{J}$-group H. Then there are only finitely many inequivalent complements of H in G and of $\hat{H}$ in $\hat{G}$. All complements of QH in QG, of $Q\hat{H}$ in $Q\hat{G}$ and of ΠH in ΠG are equivalent in their respective groups.*

Proof. We proceed by induction on the class of H. If H is abelian, the number of inequivalent complements of XH in XG is the order of the cohomology group $H^1(E, XH)$. If XH is divisible, the order of this group is one. In any case XH/mXH is a finite group for any integer m so that $H^1(E, XH)$ is finite [17, pp. 89–91].

Suppose that H is not abelian and let Z be the center of H. Z is a normal subgroup of G and G/Z is a finite split extension of the $\mathcal{J}$-group H/Z, which is of smaller class than H. By the inductive assumption there are complements $E_1XZ/XZ, \cdots, E_rXZ/XZ$ such that every complement EXZ/XZ of XH/XZ is equivalent in XG/XZ to one of these. If XH is divisible there will be only one. If EXZ/XZ is equivalent to E_iXZ/XZ, then there is an h in XH such that $h^{-1}(EXZ)h = E_iXZ$. Consider $K = E_iZ$. K is a split extension of the torsion-free abelian group Z by E_i. Consequently there are only finitely many inequivalent complements of XZ in $X(E_iZ) = E_iXZ$. Let them be $E_i = E_{i,1}, \cdots, E_{i,m(i)}$. Again there is only one if XZ is divisible. Now if E is a complement such that $h^{-1}(EXZ)h = E_iXZ$ for some h in XH, then $h^{-1}Eh$ is a complement of XZ in E_iXZ. Thus there is an element z in XZ such that $z^{-1}h^{-1}Ehz = E_{i,\lambda(i)}$ for some $\lambda(i)$ with $1 \leq \lambda(i) \leq m(i)$. This implies that every complement of XH in XG is equivalent to one of the finite set

$$E_{1,1}, \cdots, E_{1,m(1)}, \cdots, E_{r,1}, \cdots, E_{r,m(r)}.$$

As before this set is just $E_{1,1}$ if XH is divisible. This completes the proof of the lemma.

Let E be a complement of H in G. Let $\mathfrak{U}_A$ be the subgroup of $\mathcal{G}_A$ which stabilizes E and let $\mathfrak{U}_A^\infty = \mathfrak{U}_A \cap \mathcal{G}_A^\infty$ and $\mathfrak{U}_Q = \mathfrak{U}_A \cap \mathcal{G}_Q$. If f is an element of $Q\hat{H}$, then there is an integer m such that f^m is in $\hat{H} = \Pi_{p\in P} Z_pH$. Since ΠH is divisible and contains $\hat{H}$, there is a canonical embedding of $Q\hat{H}$ in ΠH which takes $\hat{H} < Q\hat{H}$ identically onto $\hat{H} < \Pi\hat{H}$. The embedding takes f to an element with coordinates in Z_pH for all primes p which do not divide m. Consequently conjugation by f stabilizes each of these Z_pH, so the induced automorphism $f^\sim$ of ΠG lies in $\mathcal{G}_A$. Let $\mathcal{F}_A$ denote the subgroup of $\mathcal{G}_A$ consisting of automorphisms given by conjugation by elements of $Q\hat{H}$. If α is in $\mathcal{G}_A$ and f is in $Q\hat{H}$, then $\alpha(f)$ is in Z_pH for all but a finite number of primes p. Thus there is an integer n (a product of powers of those primes) such that $\alpha(f)^n$ is in $\hat{H}$, i.e. $\alpha(f)$ is in $Q\hat{H}$. Clearly then $\mathcal{G}_A$ stabilizes $Q\hat{H}$. Since $\alpha f^\sim \alpha^{-1} = (\alpha(f))^\sim$, which is again in $\mathcal{F}_A$, the group $\mathcal{F}_A$ is normal in $\mathcal{G}_A$. Let $\mathcal{F}_A^\infty$ and $\mathcal{F}_Q$ denote those automorphisms of ΠG given by conjugation by elements of $\hat{H}$ and QH respectively. Let π be the canonical projection of $\mathcal{G}_A$ on $\mathcal{G}_A/\mathcal{F}_A$.

Fifth reduction. It is sufficient to show that the number of double cosets $\pi\mathfrak{U}_A^\infty \backslash \pi\mathfrak{U}_A / \pi\mathfrak{U}_Q$ is finite.

Proof. Applying Lemma 6, we see that $\mathcal{G}_A$ stabilizes equivalence classes of complements of $Q\hat{H}$ in $Q\hat{G}$, $\mathcal{G}_Q$ stabilizes equivalence classes of complements of QH in QG and there is a subgroup K of finite index in $\mathcal{G}_A^\infty$ which stabilizes the equivalence class of E in $\hat{G}$. (Any α in $\mathrm{Aut}(XG)$ takes E to some complement of XH. In the first two cases there is only one equivalence class. K is the subgroup of $\mathcal{G}_A^\infty$ which sends E to a complement in its equivalence class. Since the number of equivalence classes is finite, K is of finite index in $\mathcal{G}_A^\infty$.) Consequently for any α in $\mathcal{G}_A$ we have $\alpha(E) = f^{-1}Ef$ for some f in $Q\hat{H}$ or $\alpha \cdot (f^\sim)^{-1}$ is in $\mathfrak{U}_A$. Thus $\mathcal{G}_A = \mathfrak{U}_A\mathcal{F}_A$ and similarly $\mathcal{G}_Q = \mathfrak{U}_Q\mathcal{F}_Q$ and $K = \mathfrak{U}_A^\infty\mathcal{F}_A^\infty$. Since K is of finite index in $\mathcal{G}_A^\infty$, it is sufficient to show that the number of double cosets $\mathfrak{U}_A^\infty\mathcal{F}_A^\infty \backslash \mathfrak{U}_A\mathcal{F}_A / \mathfrak{U}_Q\mathcal{F}_Q$ is finite.

Lemma 7. *Suppose G is a group with subgroups H and K and normal subgroup N. Let the canonical projection be $\pi\colon G \longrightarrow G/N$. Suppose that N satisfies $N = x^{-1}(N \cap H)x(N \cap K)$ for any x in G. Then the number of double cosets $H\backslash G/K$ is equal to the number of double cosets $\pi H\backslash \pi G/\pi K$.*

Proof of lemma. Choose $\{x_\alpha \in G \mid \alpha \text{ in } A\}$ so that

$$\pi G = \bigcup_{\alpha \in A} (\pi H)(\pi x_\alpha)(\pi K).$$

Then

$$G = \bigcup_{\alpha\in A} H(x_\alpha N)K = \bigcup_{\alpha\in A} H(x_\alpha(x_\alpha^{-1}(N \cap H))x_\alpha(N \cap K))K$$

$$= \bigcup_{\alpha\in A} H((N \cap H)x_\alpha(N \cap K))K = \bigcup_{\alpha\in A} Hx_\alpha K.$$

Thus $|H\backslash G/K| \leq |\pi H\backslash \pi G/\pi K|$. The other inequality is obvious.

To complete the fifth reduction, we wish to apply Lemma 7 with $\mathfrak{F}_A$ in place of N, since obviously $\pi(\mathfrak{U}_A \mathfrak{F}_A) = \pi\mathfrak{U}_A$, $\pi(\mathfrak{U}_A^\infty \mathfrak{F}_A^\infty) = \pi\mathfrak{U}_A^\infty$ and $\pi(\mathfrak{U}_Q \mathfrak{F}_Q) = \pi\mathfrak{U}_Q$. We must therefore show that $\mathfrak{F}_A$ satisfies the additional hypothesis. Since $\mathfrak{U}_A^\infty\mathfrak{F}_A^\infty \cap \mathfrak{F}_A \supset \mathfrak{F}_A^\infty$ and $\mathfrak{U}_Q\mathfrak{F}_Q \cap \mathfrak{F}_A \supset \mathfrak{F}_Q$, it is sufficient to show that $\mathfrak{F}_A = \alpha^{-1}(\mathfrak{F}_A^\infty)\alpha\mathfrak{F}_Q$ for any α in $\mathfrak{G}_A$. By Lemma 4, there is a unipotent algebraic matric group $\mathfrak{H}$, such that H is isomorphic to $\mathfrak{H}_Z$ by an isomorphism which extends to take QH onto $\mathfrak{H}_Q$, Z_pH onto $\mathfrak{H}_{Z_p}$ and Q_pH onto $\mathfrak{H}_{Q_p}$. Thus we have an isomorphism from $Q\hat{H}$ onto $\mathfrak{H}_A$ (see [5, pp. 6, 7], for definitions), which takes $\hat{H}$ to $\mathfrak{H}_A^\infty$ and QH to $\mathfrak{H}_Q$. Since unipotent algebraic groups have the strong approximation property (see [5, p. 13]), if α is an automorphism of $\mathfrak{H}_A$, $\mathfrak{H}_A = \alpha(\mathfrak{H}_A^\infty)\cdot\mathfrak{H}_Q$. Thus if α is an element of $\mathfrak{G}_A$, it induces an automorphism of $Q\hat{H}$ and $Q\hat{H} = \alpha(\hat{H})\cdot QH$. Note that $\mathfrak{F}_A$ is a homomorphic image of $Q\hat{H}$ by a homomorphism ψ which sends QH to $\mathfrak{F}_Q$ for any α in $\mathfrak{G}_A$, sends $\alpha(\hat{H})$ to $\alpha\mathfrak{F}_A^\infty\alpha^{-1}$. Consequently $\mathfrak{F}_A = \alpha\mathfrak{F}_A^\infty\alpha^{-1}\mathfrak{F}_Q$ for any α in $\mathfrak{G}_A$ and the hypotheses of Lemma 7 are satisfied. This completes the fifth reduction.

Let $\mathcal{H}_A$ be the group of automorphisms of ΠH which stabilize each Q_pH and stabilize almost all Z_pH. Let $\mathcal{H}_A^\infty$ be the subgroup which stabilizes each Z_pH and let $\mathcal{H}_Q$ be the subgroup obtained by extension of automorphisms of QH. Let $\mathfrak{S}$ be the subgroup of $\mathcal{H}_A$ given by conjugation by elements of the complement E. Let $\mathfrak{M}$ be the normalizer of $\mathfrak{S}$ in $\mathrm{Aut}(\Pi H)$ and let $\mathcal{C}$ be the centralizer of $\mathfrak{S}$ in $\mathrm{Aut}(\Pi H)$. Define $\mathfrak{M}_A = \mathfrak{M} \cap \mathcal{H}_A$, $\mathfrak{M}_A^\infty = \mathfrak{M} \cap \mathcal{H}_A^\infty$, $\mathfrak{M}_Q = \mathfrak{M} \cap \mathcal{H}_Q$, $\mathcal{C}_A = \mathcal{C} \cap \mathcal{H}_A$, $\mathcal{C}_A^\infty = \mathcal{C} \cap \mathcal{H}_A^\infty$ and $\mathcal{C}_Q = \mathcal{C} \cap \mathcal{H}_Q$. If ρ is the restriction homomorphism from $\mathfrak{G}_A$ to $\mathcal{H}_A$, $\rho(\mathfrak{F}_A)$ is a normal subgroup of $\mathcal{H}_A$. We denote the canonical projection of $\mathcal{H}_A$ on $\mathcal{H}_A/\rho\mathcal{F}_A$ by π' and denote by $\bar{\rho}$ the homomorphism from $\pi\mathcal{G}_A$ to $\pi'\mathcal{H}_A$ induced by ρ.

Sixth reduction. It is sufficient to show that the number of double cosets $\pi'\mathcal{C}_A^\infty\backslash\pi'\mathcal{C}_A/\pi'\mathcal{C}_Q$ is finite.

Proof. For e in E, let $e^\sim$ denote the automorphism given by conjugation by e and let μ be in $\mathfrak{U}_A$. Since $\mathfrak{U}_A$ stabilizes E, $\mu e^\sim \mu^{-1} = (\mu(e))^\sim$ is again conjugation by an element of E. Thus we have $\rho\mathfrak{U}_A \subset \mathfrak{M}_A$, $\rho\mathfrak{U}_A^\infty \subset \mathfrak{M}_A^\infty$ and $\rho\mathfrak{U}_Q \subset \mathfrak{M}_Q$. Also since ζ in $\mathcal{C}_A$ centralizes E, we may extend ζ to ΠG by $\zeta'(eh) = e\zeta(h)$ for e in E and h in ΠH. This extension obviously stabilizes E so it lies in $\mathfrak{U}_A$ and $\rho(\zeta') = \zeta$. Consequently we have $\mathcal{C}_A \subset \rho\mathfrak{U}_A$, $\mathcal{C}_A^\infty \subset \rho\mathfrak{U}_A^\infty$ and $\mathcal{C}_Q \subset \rho\mathfrak{U}_Q$. Since E is finite, $\mathcal{C}$ is of finite index in $\mathfrak{M}$ so each of the above inclusions is of finite index.

If μ in $\mathfrak{U}_A$ represents a coset in $\pi\mathfrak{U}_A$ which is in the kernel of $\bar{\rho}$, then $\rho(\mu)$ is in $\mathfrak{M}_A \cap \rho\mathcal{F}_A$. If $\rho(\mu)$ is in $\mathcal{C}_A \cap \rho\mathcal{F}_A$, then $\rho(\mu)$ lifts back to an element μ' of $\mathfrak{U}_A \cap \mathcal{F}_A$ which differs from μ by an element in $\mathfrak{U}_A \cap \ker\rho$. Since $\mathfrak{U}_A$ stabilizes E, $\ker\bar{\rho} \cap \mathfrak{U}_A$ is contained in Aut(E) which is finite. Thus there can be only finitely many cosets in $\ker\bar{\rho}$ whose representatives are taken by ρ to $\mathcal{C}_A \cap \rho\mathcal{F}_A$. Since $\mathcal{C}_A \cap \rho\mathcal{F}_A$ is of finite index in $\mathfrak{M}_A \cap \rho\mathcal{F}_A$, and every coset in $\ker\bar{\rho}$ has its representatives mapped by ρ to $\mathfrak{M}_A \cap \rho\mathcal{F}_A$, $\ker\bar{\rho}$ must be finite. Thus it is sufficient to show that the number of double cosets $\bar{\rho}\pi\mathfrak{U}_A^\infty\backslash\bar{\rho}\pi\mathfrak{U}_A/\bar{\rho}\pi\mathfrak{U}_Q$ is finite. By the previous paragraph each of the following inclusions is of finite index: $\pi'\mathcal{C}_A \subset \bar{\rho}\pi\mathfrak{U}_A$, $\pi'\mathcal{C}_A^\infty \subset \bar{\rho}\pi\mathfrak{U}_A^\infty$ and $\pi'\mathcal{C}_Q \cap \bar{\rho}\pi\mathfrak{U}_Q$. From this the reduction follows.

Seventh reduction. It is sufficient to show that the number of double cosets $\mathcal{C}_A^\infty\backslash\mathcal{C}_A/\mathcal{C}_Q$ is finite.

Proof. We wish to apply Lemma 7 to π' restricted to $\mathcal{C}_A$, with kernel $\mathcal{C}_A \cap \rho\mathcal{F}_A = \mathcal{C} \cap \rho\mathcal{F}_A$. As before we must verify that the additional hypothesis holds. Let $Q\hat{C}$ be the centralizer of E in $Q\hat{H}$, let $\hat{C}$ be the centralizer of E in QH. If α is an element of $\mathcal{C}_A$, α stabilizes $Q\hat{H}$ since it is in H_A and since α centralizes E, α stabilizes $Q\hat{C}$. Consider the map $\psi: Q\hat{C} \to \rho\mathcal{F}_A$ by $\psi(f) = \rho(f^\sim)$, the inner automorphism of ΠH, defined by f. We then have $\psi(Q\hat{C}) = \mathcal{C} \cap \rho\mathcal{F}_A$, $\psi(QC) = \mathcal{C} \cap \rho\mathcal{F}_Q$ and $\psi(\alpha(\hat{C})) = \alpha(\mathcal{C} \cap \rho\mathcal{F}_A^\infty)\alpha^{-1}$ for all α in $\mathcal{C}_A$. The kernel of π' restricted to $\mathcal{C}_A$ is $\mathcal{C}_A \cap \rho\mathcal{F}_A$, and $(\mathcal{C}_A \cap \rho\mathcal{F}_A) \cap \mathcal{C}_A^\infty \supset \mathcal{C} \cap \rho\mathcal{F}_A^\infty$ and $(\mathcal{C}_A \cap \rho\mathcal{F}_A) \cap \mathcal{C}_Q \supset \mathcal{C} \cap \rho\mathcal{F}_Q$. Thus the additional hypothesis of Lemma 7 will be satisfied if we show that $Q\hat{C} = \alpha(\hat{C}) \cdot QC$ for any α in $\mathcal{C}_A$. If $\mathfrak{H}$ is the unipotent algebraic matric group to which H corresponds, the elements of E act algebraically on $\mathfrak{H}$. Thus the subgroup $\mathfrak{Z}$ of $\mathfrak{H}$ fixed by the elements of E is an algebraic subgroup of $\mathfrak{H}$. The correspondence of the fifth reduction, between $Q\hat{H}$ and $\mathfrak{H}_A$, sends $Q\hat{C}$ onto $\mathfrak{Z}_A$, $\hat{C}$ onto $\mathfrak{Z}_A^\infty$ and QC onto $\mathfrak{Z}_Q$. Since $\mathfrak{Z}$ is a closed subgroup of the unipotent group $\mathfrak{H}$, $\mathfrak{Z}$ is also unipotent and thus has the strong approximation property. From this it follows as before that $Q\hat{C} = \alpha(\hat{C}) \cdot QC$ for each α in $\mathcal{C}_A$ so the reduction is accomplished.

Let L be the lie algebra of H. By the fourth reduction, $\log H$ is a lattice in L. As in [14] there is an algebraic matric group $\mathfrak{A}$ and an isomorphism of Aut(H) with $\mathfrak{A}_Z$, which extends to isomorphisms which take Aut(QH) onto $\mathfrak{A}_Q$, Aut(Z_pH) onto $\mathfrak{A}_{Z_p}$, Aut(Q_pH) onto $\mathfrak{A}_{Qp}$, $\mathcal{H}_A$ onto $\mathfrak{A}_A$ and $\mathcal{H}_A^\infty$ onto $\mathfrak{A}_A^\infty$. The elements of E define automorphisms of H and so define a subgroup $\mathfrak{E}$ of $\mathfrak{A}_Z \subset \mathfrak{A}$. The centralizer of this finite set in $\mathfrak{A}$ is an algebraic matric group $\mathfrak{C}$. The above defined isomorphism of $\mathcal{H}_A$ with $\mathfrak{A}_A$ restricts to take $\mathcal{C}_A$ onto $\mathfrak{C}_A$, $\mathcal{C}_A^\infty$ onto $\mathfrak{C}_A^\infty$ and $\mathcal{C}_Q$ onto $\mathfrak{C}_Q$. Thus the number of double cosets $\mathcal{C}_A^\infty \backslash \mathcal{C}_A / \mathcal{C}_Q$ is equal to the number of double cosets $\mathfrak{C}_A^\infty \backslash \mathfrak{C}_A / \mathfrak{C}_Q$. This last set of double cosets is finite by Theorem 5.1 of [5]. This completes the proof of the main Theorem.

Appendix. Proof of Lemma 4. If $X_1, \cdots, X_n$ is a normal basis (see [14]) for H, the correspondence

$$X = X_1^{\xi(n)} \cdots X_n^{\xi(n)} \to (\xi(1), \cdots, \xi(n))$$

gives a one-to-one mapping of H onto the integer points of an n-dimensional affine algebraic variety A^n, with coordinate ring $\mathfrak{A} = k(T_1, \cdots, T_n)$. By Theorem 6.5 of [10], the product $(\bar{\zeta})$ of two n-tuples $(\bar{\xi})$ and $(\bar{\eta})$ is given by

$$\begin{aligned}\zeta(i) &= p_i(\bar{\xi}, \bar{\eta}) \\ &= \xi(i) + \eta(i) + \sum_j r_{ij}(\xi(1), \cdots, \xi(i-1)) \cdot s_{ij}(\eta(1), \cdots, \eta(i-1))\end{aligned}$$

where $p_i(\bar{\xi}, \bar{\eta})$ are polynomials which take integer values for all integer $\bar{\xi}, \bar{\eta}$. Thus we may choose r_{ij} and s_{ij} to be integer multiples of products of the form

$$r_{ij} = \binom{\xi(1)}{n(1)} \cdots \binom{\xi(i-1)}{n(i-1)} \quad \text{and} \quad s_{ij} = \binom{\eta(1)}{m(1)} \cdots \binom{\eta(i-1)}{m(i-1)}$$

where $\binom{x}{n}$ denotes the "binomial coefficient"

$$(1/n!)(X(X-1) \cdots (X-n+1)).$$

We may give A^n an algebraic group structure compatible with this multiplication by defining the comultiplication

$$\mu_0 \colon \mathfrak{A} = k[T_1, \cdots, T_n] \to \mathfrak{A} \otimes \mathfrak{A} = k[R_1 \cdots R_n] \otimes k[S_1 \cdots S_n]$$

by

$$\mu_0(T_i) = R_i \otimes 1 + 1 \otimes S_i + \sum_j r_{ij}(R_1, \cdots, R_{i-1}) \otimes S_{ij}(S_1, \cdots, S_{i-1}).$$

We now show that the isomorphism of (A^n, μ) with an algebraic matric group as given by Theorem 1.10 of [4] can be chosen so as to take (A^n, μ) to a unipotent algebraic matric group and so that integer points go onto integer points. We first observe that if

$$X(T_1, \cdots, T_n) = \binom{T_1}{m_1} \cdots \binom{T_n}{m_n}$$

then

$$(2) \qquad \mu(X) = X(R_1, \cdots, R_n) \otimes 1 + 1 \otimes X(S_1, \cdots, S_n) + \sum_j r_j(\overline{R}) \otimes s_j(\overline{S})$$

where r_i and s_j are "binomial monomials". This holds because X has integer values for all integral values ζ_i of T_i and for all integral values ξ_i of R_i and η_i of S_i, $\mu_0(T_i)(\overline{\xi}, \overline{\eta}) = \zeta_i$ is integral. Thus $\mu_0(X)$ has integral values for all integral values of R_i and S_j and so the r_j and s_j can be chosen to be "binomial monomials". We now lexicographically order the "binomial monomials" by

$$X = \binom{T_1}{m_1} \cdots \binom{T_n}{m_n} < Y = \binom{T_1}{k_1} \cdots \binom{T_n}{k_n}$$

if and only if $m_j = k_j$ for $j >$ some i and $m_i < k_i$. With this ordering it is clear from the definition of μ_0 that

$$(3) \qquad \begin{aligned} \mu_0(X) &= X(\overline{R}) \otimes 1 + 1 \otimes X(\overline{S}) \\ &\quad + \sum_j \text{ tensor products of terms less than } X \text{ in the ordering.} \end{aligned}$$

Using (2) and (3) we may choose a finite set of monomials $X(\overline{T})$ which generate A as an algebra and span a subspace of A which is invariant under right translations. Suppose this set is denoted by

$$1 = f_1, \cdots, f_r = T_n \quad (\text{with } f_{i(j)} = T_j)$$

in increasing order. Suppose g is any element of A^n. Then right translation by g yields

$$f_i\rho(g) = \sum_j f_j m_{ij}(g)$$

where by (3), $m_{ij}(g) = 0$ if $i < j$, $m_{ii}(g) = 1$ and $m_{i1}(g) = f_i(g)$. By (2) if g has integer coordinates then m_{ij} is an integer for all i and j. Conversely if $m_{ij}(g)$ is an integer for all i and j then in particular $m_{i(j),1}(g) = T_j(g)$ is an integer so that g has integer coordinates. Thus the $m_{ij}(g)$ form a unipotent algebraic matric group $\mathfrak{H}$ with integer points isomorphic to H. The extension of the isomorphism to the groups XH for $X = Q$, Z_p or Q_p follows immediately, since the same correspondence sends elements of XH onto X-valued points of A^n (Lemma 1.3 of [14]) which in turn go onto X-valued points of $\mathfrak{H}$ since each of the rings X is closed under formation of binomial coefficients.

If α is an automorphism of H, the coordinates of $\alpha(x) = \alpha(x_1)^{\xi(1)} \cdot \alpha(x_2)^{\xi(2)} \cdots \alpha(x_n)^{\xi(n)}$ are given by polynomials in $\xi(i)$ and the coordinates of $\alpha(x_i)$ by Theorem 6.5 of [10]. Thus α defines an algebraic automorphism of A^n and also an algebraic automorphism of $\mathfrak{H}$.

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